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GENERAL SOLUTION OF THE TRANSMISSION OF FORCE IN A STEAM ENGINE, INCLUDING FRICTION, ACCELERATION, AND GRAVITY.

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This problem has been discussed by many, but the author has met with no perfectly general solution. Various approximate ones are employed in general engineering work involving errors of unknown magnitude. The author has, therefore, endeavored in this paper to present a general set of equations by means of which the errors of the more approximate ones may be ascertained.

The following general conditions have been assumed:

(*a*) that the centre of the crank shaft is not necessarily on the line of travel of the wrist-pin;†

(*b*) that the centre of gravity of the connecting-rod is not necessarily in its line of centres;

(*c*) that the crank revolves at a uniform speed;

(*d*) that the mass of the moving parts is distributed in any manner;

(*e*) that the line of travel of the wrist pin is not necessarily horizontal or vertical;

(*f*) that there is friction between the parts.

Fig. 1 represents the main lines of an engine referred to the horizontal axis OX, coinciding with the line of travel of the wrist-pin, O being the extreme point of its travel and OY the vertical axis; the directions of the various quantities are indicated by arrow-heads.

NOTATION.

Let $F_1 =$ the force to accelerate the piston, piston-rod, and cross-head;

* I am greatly indebted to Prof. J. Burkitt Webb for valuable assistance in preparing this article, many of the equations having been worked out independently by him, and compared in order to check the final result. The methods employed in the solutions have also been in a great measure suggested by him.

† Engines are now in use that require this condition to enter the equations; for example, the Westinghouse.

X_1, X_2 and Y_1, Y_2 = the x and y components of the forces which accelerate the connecting-rod, applied at the wrist and crank-pins respectively;

R = radius of the crank;

nR = length of the line joining the wrist and crank-pin centres;

lR = distance from the wrist-pin to the foot of the perpendicular let fall from the centre of gravity of the connecting-rod upon this line;

cR = length of this perpendicular;

bR = perpendicular from the centre of the crank shaft upon the line of motion of the wrist-pin;

θ = crank angle;

β = connecting-rod angle;

M = mass of piston, piston-rod, and cross-head;

m = mass of the connecting-rod, and

kR = its principal radius of gyration;

W_1 = weight of the piston, piston-rod, and cross-head;

W_2 = weight of the connecting-rod;

τ = angular velocity of crank;

s = length of stroke;

t = time required by the crank to turn through the angle θ.

FORCE REQUIRED TO ACCELERATE THE PISTON, PISTON-ROD, AND CROSS-HEAD.

The general value of x, the co-ordinate of the wrist-pin, is

$$x = \sqrt{(R + nR)^2 - b^2R^2} - R(\cos\theta + n\cos\beta). \tag{1}$$

From the figure we have

$$\sin\beta = \frac{\sin\theta - b}{n}; \tag{2}$$

therefore,

$$\cos\beta = \frac{1}{n}\sqrt{n^2 - (\sin\theta - b)^2}, \tag{3}$$

which reduces (1) to

$$x = \sqrt{(R + nR)^2 - b^2R^2} - R[\cos\theta + \sqrt{n^2 - (\sin\theta - b)^2}]. \tag{4}$$

Differentiating, we have

$$\frac{dx}{d\theta} = R\left(\sin\theta + \frac{(\sin\theta - b)\cos\theta}{\sqrt{n^2 - (\sin\theta - b)^2}}\right),$$

and

$$\frac{d^2x}{d\theta^2} = R\left(\cos\theta - \frac{\sin\theta(\sin\theta - b)}{\sqrt{n^2 - (\sin\theta - b)^2}} + \frac{n^2\cos^2\theta}{[n^2 - (\sin\theta - b)^2]^{\frac{3}{2}}}\right).$$

For brevity, let

$$\frac{n^2 \cos^2\theta}{[n^2 - (\sin \theta - b)^2]^{\frac{3}{2}}} - \frac{\sin \theta (\sin \theta - b)}{\sqrt{n^2 - (\sin \theta - b)^2}} = Z; \tag{5}$$

then

$$\frac{d^2x}{d\theta^2} = R (\cos \theta + Z). \tag{6}$$

But

$$\tau = \frac{d\theta}{dt}, \tag{7}$$

by means of which $d\theta$ may be eliminated from (6), giving

$$\frac{d^2x}{dt^2} = \tau^2 R (\cos \theta + Z);$$

from which it follows that

$$F_1 = M\tau^2 R (\cos \theta + Z). \tag{8}$$

FORCES REQUIRED TO ACCELERATE THE CONNECTING-ROD.

The general equations for the forces to accelerate the connecting-rod have been derived by two methods:

1. By determining the forces at the centre of gravity required to vary the translation of the mass, and the moment necessary for its angular acceleration;

2. By assuming the mass of the rod to be symmetrically arranged, say in two equal and opposite portions, at a distance from the centre of gravity equal to the radius of gyration, and determining the forces and moments required for the acceleration of such a representative rod.

FIRST METHOD.

The motion of the connecting-rod is composed of a translation of its centre of gravity and a rotation of the rod about it, this translation and rotation being so related as to cause the wrist pin to follow a right line, and the crank pin a circle. This motion might be considered as a rectilinear translation of the wrist pin and a rotation about it, but is not so taken, because the centrifugal forces of the parts of the rod are not balanced about that point and complicate the problem.

To determine the forces at the wrist and crank pins required to accelerate

the centre of gravity, we determine the force necessary when applied at that point, and resolve it into two equivalent forces applied at the wrist and crank pins, which latter forces will be inversely proportional to their lever arms about the centre of gravity. We thus obtain forces at the pins that will produce translation only.

To determine the rotative forces we find the moment necessary to produce the rotation, and then place such equal and opposite forces at the pins as will produce it.

The sum of the translative and rotative forces at the pins will be the total forces required to produce the complete acceleration of the rod.

Let x and y be the co-ordinates of the centre of gravity; (see Fig. 1) then we have

$$x = \sqrt{(R + nR)^2 - b^2R^2} - R\,[\cos\theta + (n - l)\cos\beta + c\sin\beta].$$

Substituting the values of $\sin\beta$ and $\cos\beta$ from (2) and (3), we obtain

$$x = \sqrt{(R + nR)^2 - b^2R^2} - R\left\{\cos\theta + \frac{n-l}{n}\sqrt{n^2 - (\sin\beta - b)^2} + \frac{c\,(\sin\theta - b)}{n}\right\};$$

from which

$$\frac{dx}{d\theta} = R\left\{\sin\theta + \frac{(n-l)\,(\sin\theta - b)\cos\theta}{n\sqrt{n^2 - (\sin\theta - b)^2}} - \frac{c\cos\theta}{n}\right\},$$

and

$$\frac{d^2x}{d\theta^2} = R\left\{\cos\theta - \frac{(n-l)\,(\sin\theta - b)\sin\theta}{n\sqrt{n^2 - (\sin\theta - b)^2}} + \frac{(n-l)\,n\cos\theta}{[n^2 - (\sin\theta - b)^2]^{\frac{3}{2}}} + \frac{c\sin\theta}{n}\right\}.$$

Introducing the values of Z and t from (5) and (7), we have

$$\frac{d^2x}{dt^2} = \tau^2R\left\{\cos\theta + \frac{n-l}{n}Z + \frac{c}{n}\sin\theta\right\}.$$

Let F_x be the x component of the translative force at the centre of gravity; then

$$F_x = m\tau^2R\left\{\cos\theta + \frac{n-l}{n}Z + \frac{c}{n}\sin\theta\right\}. \tag{9}$$

From Fig. 1 we have for the ordinate of the centre of gravity

$$y = lR\sin\beta + cR\cos\beta = \frac{lR}{n}(\sin\theta - b) + \frac{cR}{n}\sqrt{n^2 - (\sin\theta - b)^2}.$$

Differentiating, we have

$$\frac{dy}{d\theta} = R\left(\frac{l}{n}\cos\theta - \frac{c(\sin\theta - b)\cos\theta}{n\sqrt{n^2 - (\sin\theta - b)^2}}\right),$$

and

$$\frac{d^2y}{d\theta^2} = -R\left(\frac{l}{n}\sin\theta - \frac{c(\sin\theta - b)\sin\theta}{n\sqrt{n^2 - (\sin\theta - b)^2}} + \frac{cn\cos^2\theta}{[n^2 - (\sin\theta - b)^2]^{\frac{3}{2}}}\right);$$

from which

$$\frac{d^2y}{dt^2} = -\tau^2 R\left(\frac{l}{n}\sin\theta + \frac{c}{n}Z\right).$$

Let F_y be the Y component of the translative force at the centre of gravity; then

$$F_y = -m\tau^2 R\left(\frac{l}{n}\sin\theta + \frac{c}{n}Z\right). \tag{10}$$

In order to determine the rotative forces, we proceed as follows:
From (2) we have

$$\frac{d\beta}{d\theta} = \frac{\cos\theta}{n\cos\beta},$$

$$\frac{d^2\beta}{d\theta^2} = -\frac{\sin\theta}{\sqrt{n^2 - (\sin\theta - b)^2}} + \frac{\cos^2\theta(\sin\theta - b)}{[n^2 - (\sin\theta - b)^2]^{\frac{3}{2}}};$$

from which it follows that the moment required to produce the rotation will be

$$M_r = -m\tau^2 R^2 k^2\left(\frac{\sin\theta}{\sqrt{n^2 - (\sin\theta - b)^2}} - \frac{\cos^2\theta(\sin\theta - b)}{[n^2 - (\sin\theta - b)^2]^{\frac{3}{2}}}\right). \tag{11}$$

Let D_w, D_c and E_w, E_c be the horizontal and vertical forces acting at the wrist and crank pins, respectively, into which the forces F_x and F_y are divided; then we have

$$D_w nR\sin\beta = F_x[(n - l)R\sin\beta - cR\cos\beta],$$

and

$$D_c nR\sin\beta = F_x(lR\sin\beta + cR\cos\beta);$$

from which

$$D_w = F_x\left(\frac{n - l}{n} - \frac{c}{n}\cot\beta\right) = F_x\left(\frac{n - l}{n} - \frac{c\sqrt{n^2 - (\sin\theta - b)^2}}{n(\sin\theta - b)}\right), \tag{12}$$

and

$$D_c = F_x\left(\frac{l}{n} + \frac{c}{n}\cot\beta\right) = F_x\left(\frac{l}{n} + \frac{c\sqrt{n^2 - (\sin\theta - b)^2}}{n(\sin\theta - b)}\right). \tag{13}$$

Dividing the vertical force F_y between the wrist and crank pins in the same way, we have

$$F_y\,[(n-l)\,R\cos\beta + cR\sin\beta] = E_w nR\cos\beta,$$

and

$$F_y\,(lR\cos\beta - cR\sin\beta) = E_c nR\cos\beta\,;$$

from which

$$E_w = F_y\left(\frac{n-l}{n} + \frac{c\,(\sin\theta - b)}{n\sqrt{n^2-(\sin\theta-b)^2}}\right), \tag{14}$$

and

$$E_c = F_y\left(\frac{l}{n} - \frac{c\,(\sin\theta - b)}{n\sqrt{n^2-(\sin\theta-b)^2}}\right). \tag{15}$$

The value of M_r given in (11) may be written

$$M_r = m\tau^2R^2\frac{k^2}{n^2}Zn\sin\beta - m\tau^2R^2\frac{k^2}{n^2}\sin\theta\,.\,n\cos\beta\,;$$

therefore the rotation may be produced by two equal and opposite forces in the X direction at the wrist and crank pins equal to $\pm\, m\tau^2R\frac{k^2}{n^2}Z$, together with a second set applied in the y direction at the same points equal to $\pm\, m\tau^2R\frac{k^2}{n^2}\sin\theta$.*

Adding these forces to those required to produce the translation, we have the total accelerating forces,

$$Y_1 = E_w + m\tau^2R\frac{k^2}{n^2}\sin\theta\,, \tag{16}$$

$$Y_2 = E_c - m\tau^2R\frac{k^2}{n^2}\sin\theta\,, \tag{17}$$

$$X_1 = D_w + m\tau^2R\frac{k^2}{n^2}Z, \tag{18}$$

$$X_2 = D_c - m\tau^2R\frac{k^2}{n^2}Z. \tag{19}$$

* This division of M_r reduces the values of the accelerating forces to a form in which the value of Z may be employed and numerical computations thus facilitated.

Substituting the values of E_w, E_c, D_w and D_c as given in equations (12)-(15), in equations (16)-(19) the accelerating forces become

$$Y_1 = -m\tau^2 R\left[\left(\frac{n-l}{n} + \frac{c(\sin\theta - b)}{n\sqrt{n^2-(\sin\theta-b)^2}}\right)\left(\frac{l}{n}\sin\theta + \frac{c}{n}Z\right) - \frac{k^2}{n^2}\sin\theta\right], \quad (20)$$

$$Y_2 = -m\tau^2 R\left[\left(\frac{l}{n} - \frac{c(\sin\theta - b)}{n\sqrt{n^2-(\sin\theta-b)^2}}\right)\left(\frac{l}{n}\sin\theta + \frac{c}{n}Z\right) + \frac{k^2}{n^2}\sin\theta\right], \quad (21)$$

$$X_1 = m\tau^2 R\left[\left(\frac{n-l}{n} - \frac{c\sqrt{n^2-(\sin\theta-b)^2}}{n(\sin\theta-b)}\right)\left(\cos\theta + \frac{n-l}{n}Z + \frac{c}{n}\sin\theta\right) + \frac{k^2}{n^2}Z\right], \quad (22)$$

$$X_2 = m\tau^2 R\left[\left(\frac{l}{n} + \frac{c\sqrt{n^2-(\sin\theta-b)^2}}{n(\sin\theta-b)}\right)\left(\cos\theta + \frac{n-l}{n}Z + \frac{c}{n}\sin\theta\right) - \frac{k^2}{n^2}Z\right]. \quad (23)$$

SECOND METHOD.

Fig. 2 represents the main lines of an engine in accordance with the second supposition; kR, kR are two radii of gyration drawn from the centre of gravity parallel to the line AB joining the centres of the crank and wrist pins, and the mass is supposed to be concentrated in two equal portions at their extremities D and E. This reduces the consideration of the motion of the rod to the more simple discussion of that of the two concentrated masses.

Let the co-ordinates of the masses D and E be x_1, y_1 and x_2, y_2, respectively then we have

$$x_1 = C_1 - R\left[\cos\theta + (n-l+k)\cos\beta + c\sin\beta\right],$$

in which C_1 is the distance from the centre O to the ordinate of D when the crank is on the inner dead centre.

Differentiating the above value of x_1, we have

$$\frac{d^2x_1}{d\theta^2} = R\left[\cos\theta + (n-l+k)\left(\cos\beta\frac{d\beta^2}{d\theta^2} + \sin\beta\frac{d^2\beta}{d\theta^2}\right) + c\left(\sin\beta\frac{d\beta^2}{d\theta^2} - \cos\beta\frac{d^2\beta}{d\theta^2}\right)\right]. \quad (24)$$

It also follows from Fig. 2 that

$$\sin\theta - b = n \sin\beta,$$

from which

$$\frac{d\beta}{d\theta} = \frac{\cos\theta}{\sqrt{n^2 - (\sin\theta - b)^2}}, \tag{25}$$

and

$$\frac{d^2\beta}{d\theta^2} = \frac{\cos^2\theta\,(\sin\theta - b)}{[n^2 - (\sin\theta - b)^2]^{\frac{3}{2}}} - \frac{\sin\theta}{\sqrt{n^2 - (\sin\theta - b)^2}}. \tag{26}$$

Substituting the values of $\frac{d\beta}{d\theta}$ and $\frac{d^2\beta}{d\theta^2}$ in (24), we have

$$\frac{d^2x_1}{d\theta^2} = R\left[\cos\theta + \frac{n-l+k}{n}\left\{\frac{n^2\cos^2\theta}{[n^2 - (\sin\theta - b)^2]^{\frac{3}{2}}} - \frac{\sin\theta\,(\sin\theta - b)}{\sqrt{n^2 - (\sin\theta - b)^2}}\right\} + \frac{c\sin\theta}{n}\right],$$

and on introducing the value of Z, given in (5), this becomes

$$\frac{d^2x_1}{d\theta^2} = R\left\{\cos\theta + \frac{n-l+k}{n}Z + \frac{c\sin\theta}{n}\right\}.$$

Let F_{dx} be the force required to produce this acceleration, and we have

$$F_{dx} = \frac{1}{2}m\tau^2R\left\{\cos\theta + \frac{n-l+k}{n}Z + \frac{c\sin\theta}{n}\right\}. \tag{27}$$

Dividing this between the crank and wrist pins gives for the portions acting at the crank pin A

$$F_{dx}A = \frac{1}{2}m\tau^2R\left[\frac{l-k+c\cot\beta}{n}\left\{\cos\theta + \frac{n-l+k}{n}Z + \frac{c\sin\theta}{n}\right\}\right]. \tag{28}$$

The force required to produce the X acceleration of the mass E, obtained in a similar way to F_{dx}, is

$$F_{ex} = \frac{1}{2}m\tau^2R\left\{\cos\theta + \frac{n-l-k}{n}Z + \frac{c\sin\theta}{n}\right\}, \tag{29}$$

and its component acting at the crank pin will be

$$F_{ex}A = \frac{1}{2}m\tau^2R\left[\frac{l+k+c\cot\beta}{n}\left\{\cos\theta + \frac{n-l-k}{n}Z + \frac{c\sin\theta}{n}\right\}\right]. \tag{30}$$

Adding these two components as given in equations (28) and (30), we have the total accelerating force X_2 at the crank pin, or

$$F_{dx}A + F_{ex}A = X_2;$$

from which

$$X_2 = m\tau^2 R\left[\left(\frac{l}{n} + \frac{c\sqrt{n^2 - (\sin\theta - b)^2}}{n(\sin\theta - b)}\right)\left(\cos\theta + \frac{n-l}{n}Z + \frac{c\sin\theta}{n}\right) - \frac{k^2}{n^2}Z\right]. \quad (31)$$

The accelerating force X_1 acting at the wrist pin may be found by obtaining the sum of the forces in the X direction at the masses D and E and subtracting X_2, or

$$X_1 = F_{dx} + F_{ex} - X_2;$$

from which

$$X_1 = m\tau^2 R\left(\cos\theta + \frac{n-l}{n}Z + \frac{c}{n}\sin\theta\right) - X_2$$

$$= m\tau^2 R\left[\left(\frac{n-l}{n} - \frac{c\sqrt{n^2 - (\sin\theta - b)^2}}{n(\sin\theta - b)}\right)\left(\cos\theta + \frac{n-l}{n}Z + \frac{c\sin\theta}{n}\right) + \frac{k^2}{n^2}Z\right]. \quad (32)$$

To determine the accelerating forces in the direction of the axis of Y we have, for the mass at D,

$$y_1 = R[(l-k)\sin\beta + c\cos\beta];$$

from which

$$\frac{d^2y_1}{d\theta^2} = -R\left(\frac{l-k}{n}\sin\theta + \frac{c}{n}Z\right). \quad (33)$$

Similarly, for the mass at E,

$$\frac{d^2y_2}{d\theta^2} = -R\left(\frac{l+k}{n}\sin\theta + \frac{c}{n}Z\right). \quad (34)$$

If F_{dy} and F_{ey} be the Y accelerating forces for the masses D and E respectively, we have from equations (33) and (34)

$$F_{dy} = -\frac{1}{2}m\tau^2 R\left(\frac{l-k}{n}\sin\theta + \frac{c}{n}Z\right), \quad (35)$$

and

$$F_{ey} = -\frac{1}{2}m\tau^2 R\left(\frac{l+k}{n}\sin\theta + \frac{c}{n}Z\right). \quad (36)$$

The component of F_{dy} acting at the crank pin will be

$$F_{dy}A = \frac{l - k - c \tan \beta}{n} F_{dy}$$
$$= -\frac{1}{2} m\tau^2 R\left[\frac{l - k - c \tan \beta}{n} \left(\frac{l - k}{n} \sin \theta + \frac{c}{n} Z\right)\right]; \tag{37}$$

and of F_{ey},

$$F_{ey}A = \frac{l + k - c \tan \beta}{n} F_{ey}$$
$$= -\frac{1}{2} m\tau^2 R\left[\frac{l + k - c \tan \beta}{n} \left(\frac{l + k}{n} \sin \theta + \frac{c}{n} Z\right)\right]. \tag{38}$$

The Y component of the accelerating force acting at the crank pin will be equal to

$$F_{dy}A + F_{ey}A,$$

or

$$Y_2 = F_{dy}A + F_{ey}A$$
$$= -m\tau^2 R\left[\left(\frac{l}{n} - \frac{c(\sin \theta - b)}{n\sqrt{n^2 - (\sin \theta - b)^2}}\right)\left(\frac{l}{n} \sin \theta + \frac{c}{n} Z\right) + \frac{k^2}{n^2} \sin \theta\right]. \tag{39}$$

The accelerating force Y_1 acting at the wrist pin may be found by taking the sum of the forces in the Y direction that accelerate the masses D and E as given in (35) and (36) and subtracting Y_2, or

$$Y_1 = F_{dy} + F_{ey} - Y_2$$
$$= -m\tau^2 R\left(\frac{l}{n} \sin \theta + \frac{c}{n} Z\right) - Y_2$$
$$= -m\tau^2 R\left[\left(\frac{n - l}{n} + \frac{c(\sin \theta - b)}{n\sqrt{n^2 - (\sin \theta - b)^2}}\right)\left(\frac{l}{n} \sin \theta + \frac{c}{n} Z\right) - \frac{k^2}{n^2} \sin \theta\right]. \tag{40}$$

Equations (40), (39), (32), and (31) are the same as (20), (21), (22), and (23); the first and second method therefore give the same results.

RATIO BETWEEN THE LENGTH OF STROKE OF THE PISTON AND THE RADIUS OF THE CRANK.

The length of stroke s will be the distance traveled by the wrist pin between the two positions at which $\theta = -\beta$,

or
$$s = R\sqrt{(n+1)^2 - b^2} - R\sqrt{(n-1)^2 - b^2}. \tag{41}$$

DETERMINATION OF THE NET FORCES ACTING AT VARIOUS POINTS.

The forces concerned in the action of an engine are the force of the steam on the piston, the forces required to accelerate the parts, and their weight and friction. We will first determine the effect of the pressure of the steam combined with the accelerating forces, omitting the weight and friction of the parts.

Let P_a = effort of the steam on the piston;
G = reaction of cross-head guides;
P_w = force exerted by the connecting rod upon the wrist pin;
P_c = force exerted by the connecting rod upon the crank pin;
R_1 = resultant in direction of the centre line of the rod;*
T = tangential component of P_c;
N = radial component of P_c;
H_c = force exerted by the bed on the crank shaft.

The force exerted by the steam on the piston is equal to the difference of the forces exerted on the two cylinder heads, or to the force acting on the cylinder as a whole; its equilibrium may, therefore, be represented by two lines representing the pressure in magnitude and direction and directly opposed to each other, as shown in Fig. 4, the resultant of which will be zero.

The equilibrium of the piston, piston-rod, and cross-head (see P, Fig. 3) is represented in Fig. 5.

From the pressure of the steam on the piston there must be subtracted the force F_1 required to accelerate the mass of the piston, piston-rod, and

* In our analysis we have determined the forces at the wrist and crank pins that will be equivalent in accelerating effect to the summation of the elementary accelerating forces acting at all portions of the mass of the rod, but these components cannot be used to obtain the internal strains at each portion of the rod, unless the law of the forces acting at the separate elements of its mass is included in the problem. R_1 does not, therefore, represent the actual force along the line of centres of the rod, but simply the resultant in that direction that must be common to the polygons of equilibrium at each of the pins. R_1 will have a variety of values, according to the choice of the forces to give the accelerating moment; but it is a principle in mechanics that the external forces cannot be altered by this means.

cross-head, in order to determine the force in the direction of the motion of the piston that acts on the wrist pin; let this force be F, and it follows that

$$F = P_a - F_1. \tag{42}$$

The equilibrium of the wrist pin (see Q, Fig. 3) is represented in Fig. 6. The forces transmitted from the cross-head to the wrist pin are the horizontal force F and the reaction of the guides G, and these are in equilibrium with the force P_w.

From Fig. 6 it follows that

$$P_w = \sqrt{(P_a - F_1)^2 + G^2}. \tag{43}$$

Let the angle made by P_w with the axis of X be γ; then

$$P_w = F \sec \gamma = (P_a - F_1) \sec \gamma. \tag{44}$$

The equilibrium of the wrist-pin box (see q, Fig. 3) is represented in Fig. 7. The force P_w is in equilibrium with the force R_1 in the direction of the line of centres of the connecting rod and the accelerating force at the wrist pin. The accelerating force at the wrist pin is the resultant of its two components X_1 and Y_1, and these components, and not the force itself, are represented in the figure.

From Fig. 7 we have

$$R_1 = (P_w \cos \gamma - X_1) \sec \beta. \tag{45}$$

The equilibrium of the crank-pin box (see s, Fig. 3) is represented in Fig. 8. X_2 and Y_2, together with R_1, are in equilibrium with the pressure P_c. From the figure we have

$$\begin{aligned} P_c &= \sqrt{[(R_1 \cos \beta - X_2)^2 + (R_1 \sin \beta - Y_2)^2]} \qquad (46) \\ &= \sqrt{\{(P_a - F_1 - X_1 - X_2)^2 + [(P_a - F_1 - X_1) \tan \beta - Y_2]^2\}}. \end{aligned}$$

Let the angle made by P_c with the axis of X be ω; then

$$P_c \cos \omega = R_1 \cos \beta - X_2. \tag{47}$$

The equilibrium of the crank pin (see S, Fig. 3) is represented in Fig. 9.

From Fig. 9 we have

$$\begin{aligned} T &= P_c \cos (90 - \theta + \omega) \\ &= P_c (\sin \theta \cos \omega + \cos \theta \sin \omega). \end{aligned}$$

From Fig. 8

$$P_c \sin \omega = R_1 \sin \beta - Y_2. \tag{48}$$

On introducing the values given in (47), (48), (45), and (44), in the above equation for T, we have

$$T = (P_a - F_1 - X_1 - X_2) \sin \theta - Y_2 \cos \theta + (P_a - F_1 - X_1) \tan \beta \cos \theta;$$

from which

$$T = (P_a - F_1 - X_1) \sec \beta \sin (\theta + \beta) - Y_2 \cos \theta - X_2 \sin \theta. \tag{49}$$

We may obtain, in a similar manner,

$$N = (P_a - F - X_1) \sec \beta \cos (\theta + \beta) + Y_2 \sin \theta - X_2 \cos \theta. \tag{50}$$

It also follows from Fig. 9 that

$$P_c = \sqrt{T^2 + N^2}. \tag{51}$$

The equilibrium of the crank shaft is represented in Fig. 10. The force P_c acting on the crank pin will be transmitted to the crank shaft, and this force will be in equilibrium with the pressure on the crank-shaft bearings H_c, and the weight of the fly-wheel, and the force exerted on the shaft by the mechanism that transmits the power developed by the engine, the sum of which we will call H_1. Let φ be the angle made by H_1 with the axis of X, and μ that made by H_c; then from Fig. 10

$$H_c \cos \mu = P_c \cos \omega + H_1 \cos \varphi.$$

Introducing values given in (48) and (45), this becomes

$$H_c = (P_a - F - X_1 - X_2) \sec \mu + H_1 \frac{\cos \varphi}{\cos \mu}. \tag{52}$$

The equilibrium of the bed of the engine (see B, Fig. 3) is represented in Fig. 11. The forces acting on the bed are the force P_a at the cylinder head, the reaction of the guides G, the force H_c at the crank-shaft journals, and the weight of the bed, which we will call B_w, acting at the angle δ with the axis of Y. These will be in equilibrium with the force between the foundation and the bed of the engine. Let this force be H_b, and the angle it makes with the axis of X be ζ; then it follows from Fig. 11 that

$$H_b \cos \zeta = P_a - H_c \cos \mu - B_w \sin \delta;$$

from which

$$H_b = (P_a - H_c \cos \mu - B_w \sin \delta) \sec \zeta. \tag{53}$$

The following value of H_b may also be obtained:

$$H_b = \sqrt{[(F_1 + X_1 + X_2 - H_1 \cos\varphi - B_w \sin\delta)^2 + (B_w \cos\delta + G + H_c \sin\mu)^2]}. \quad (54)$$

From Fig. 10 we have

$$H_c \sin\mu = H_1 \sin\varphi - P_c \sin\omega,$$

which substituted in equation (53) gives

$$H_b = \sqrt{[(F_1 + X_1 + X_2 - H_1 \cos\varphi - B_w \sin\delta)^2 + (B_w \cos\delta + G + H_1 \sin\varphi - P_c \sin\omega)^2]}.$$

From Figs. 5, 7, and 8 we obtain

$$P_c \sin\omega = G - Y_1 - Y_2,$$

which reduces the above value of H_b to

$$H_b = \sqrt{[(F_1 + X_1 + X_2 - H_1 \cos\varphi - B_w \sin\delta)^2 + (H_1 \sin\varphi + B_w \cos\delta + Y_1 + Y_2)^2]}. \quad (55)$$

The X component of H_b, which we will call H_{bx}, is

$$H_{bx} = H_b \cos\zeta.$$

Substituting the value of H_b as given in (53), this becomes

$$H_{bx} = F_1 + X_1 + X_2 - H_1 \cos\varphi - B_w \sin\delta.$$

It follows from (55) that the Y component will be

$$H_{by} = Y_1 + Y_2 + H_1 \sin\varphi + B_w \cos\delta.$$

If X_t and Y_t are the components of the translative force tending to shake the bed, they will act in an opposite direction to the forces that act from the foundation to the bed. As a constant force does not tend to produce a shake, X_t and Y_t will only contain that portion of H_{bx} and H_{by} that is variable. The weight of the foundation B_w will be constant, and, provided the method of transmitting power from the engine does not alter the value of H_1, which is ordinarily very nearly so, its value may also be considered as constant, and we will have

$$X_t = -F_1 - X_1 - X_2, \quad (56)$$

and

$$Y_t = -Y_1 - Y_2. \quad (57)$$

These forces, as has already been stated, tend to translate the bed of the engine, and do not indicate the value of the moment that may be produced by forces opposite in direction and not having the same line of action.

Figs. 4–11 may be combined, and a polygon formed that represents the equilibrium of the entire engine, as shown in Fig. 12.

All the equations already given for the values of the forces may be derived from Fig. 12.

Equation (49) may be written

$$T = \left[P_a - \left(F_1 + X_1 + \frac{Y_2 \cos\theta + X_2 \sin\theta}{\sec\beta \sin(\theta + \beta)} \right) \right] \sec\beta \sin(\theta + \beta).$$

The expression

$$F_1 + X_1 + \frac{Y_2 \cos\theta + X_2 \sin\theta}{\sec\beta \sin(\theta + \beta)},$$

which we will call P_b, therefore represents a pressure which on being subtracted from P_a will give a force that may be resolved as if no motion were present in order to obtain T, or

$$T = (P_a - P_b) \sec\beta \sin(\theta + \beta). \tag{58}$$

P_b may be laid off on the indicator diagram, and its value thus subtracted directly from the steam pressure.

FRICTION AND GRAVITY INCLUDED.

It has been demonstrated, in a paper prepared by Professor Webb and the writer,* that the effect of the friction at the connecting-rod bearings may be determined by introducing into the equilibrium polygons two forces, A and B, at right angles to the centre line of the rod, the sum of A and B being applied at each of the pins in opposite directions. The values of A and B are

$$A = \frac{P_{wf} r_w \sin\varphi_w}{nR}, \quad \text{and} \quad B = \frac{P_{cf} r_c \sin\varphi_c}{nR},$$

in which r_w and r_c are equal, respectively, to the radii of the wrist and crank, and $\sin\varphi_w$ and $\sin\varphi_c$ to the sines of the angles of friction. The effect of the weight was also included, and equilibrium polygons given showing the relation of the forces at the wrist and crank pins, together with equations deduced from the same.

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